$$\begin{array}{r|l} & 1 \\ \hline x^2+3x+2 & x^2+6 \\ & -(x^2+3x+2) \\ \hline & -3x+4 \end{array}$$

Entry Task: Evaluate

$$\int \frac{x^2+6}{x^2+3x+2}dx$$

$$= \int 1 + \frac{-3x+4}{x^2+3x+2}dx$$

$$\frac{-3x+4}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$-3x+4 = A(x+2) + B(x+1)$$

$$x=-1 \Rightarrow 7 = A$$

$$x=-2 \Rightarrow 10 = B(-1) \quad B = -10$$

CHECK : $-3x+4 = (A+B)x + (2A+B)$

$A+B = -3$ ✓

$2A+B = 4$ ✓

$$= \int 1 + \frac{7}{x+1} - \frac{10}{x+2}$$

$$= \boxed{x + 7\ln|x+1| - 10\ln|x+2| + C}$$

CHECK:

$$1 + \frac{7}{x+1} - \frac{10}{x+2}$$

Simplify
or plug in values

$x=0$ $\quad 1 + 7 - \frac{10}{2} = 3$ $\qquad \frac{0^2+6}{0^2+3(0)+2} = 3$

Example: Evaluate

$$\int \frac{x^2-x+6}{x^3+3x}dx$$

$$\frac{x^2-x+6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

$$\Rightarrow \quad x^2-x+6 = A(x^2+3)+(Bx+C)x$$

$$= Ax^2+3A+Bx^2+Cx$$

$$x^2-x+6 = (A+B)x^2+Cx+3A$$

$$\Rightarrow \quad A+B=1 \Rightarrow B=1-A$$

$$C=-1$$

$$3A=6 \Rightarrow A=2$$

$$B=1-A \Rightarrow B=-1$$

$$\int \frac{2}{x} + \frac{-x-1}{x^2+3}dx$$

$$= \int \frac{2}{x} + \frac{-x}{x^2+3} + \frac{-1}{x^2+3}dx$$

$$= 2\ln|x| - \int \frac{x}{x^2+3}dx + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

$$u = x^2+3$$

$$du = 2x\,dx$$

$$\frac{1}{2x}du = dx$$

$$\int \frac{\cancel{x}}{u}\frac{1}{2\cancel{x}}du$$

$$\frac{1}{2}\ln|u|$$

$$= \boxed{2\ln|x| - \frac{1}{2}\ln|x^2+3| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C}$$

Example:

$$\int \frac{x}{x^2+4x+5}\,dx$$

IRREDUCIBLE!

$x^2+4x+5 \overset{?}{=} 0$ HAS NO REAL SOL'NS

$$x = \frac{-4 \pm \sqrt{16-4(5)}}{2}$$

-4

$$\underbrace{x^2+4x+4}-4+5$$

$$(x+2)^2+1$$

CHECK!

$$\int \frac{x}{(x+2)^2+1}\,dx$$

$t = x+2 \qquad x = t-2$

$dt = dx$

$$\int \frac{t-2}{t^2+1}\,dt$$

$$\int \frac{t}{t^2+1}\,dt - \int \frac{2}{t^2+1}\,dt$$

$u = t^2+1$

$du = 2t\,dt$

$\frac{1}{2t}du = dt$

$$\int \frac{\cancel{t}}{u}\frac{1}{2\cancel{t}}\,du - 2\tan^{-1}(t) + C$$

$$\frac{1}{2}\ln|u|$$

$$= \frac{1}{2}\ln|t^2+1| - 2\tan^{-1}(t) + C$$

$$= \boxed{\frac{1}{2}\ln\left((x+2)^2+1\right) - 2\tan^{-1}(x+2) + C}$$

How to integrate

A. Look for simplifications/substitutions

B. Products/Logs/Inverse Trig → BY PARTS

Sin/Cos/Tan/Sec combos → TRIG

Quadratic (under a radical) → TRIG SUB

Rational Function → PART. FRAC.

C. If nothing seems to work, substitution.

(u = inside, u = $\sqrt{}$, u = trig, u = e^x)

Examples:

1. $\int e^{\sqrt{x}}dx$

$t = \sqrt{x} \Rightarrow t^2 = x$

$2t\,dt = dx$

$\int e^{t}\,2t\,dt$

$\int 2t\,e^t\,dt$

$u = 2t \quad dv = e^t dt$

$du = 2dt \quad v = e^t$

$= 2te^t - \int 2e^t dt$

$= 2te^t - 2e^t + C$

$= \boxed{2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C}$

2. $\int \frac{\cos(x)}{4-\sin^2(x)}dx$

$t = \sin(x)$

$dt = \cos(x)dx$

$$\int \frac{1}{4-t^2}dt$$

$$\frac{1}{(2-t)(2+t)} = \frac{A}{2-t} + \frac{B}{2+t}$$

$1 = A(2+t) + B(2-t)$

$t=2 \Rightarrow A = 1/4$

$t=-2 \Rightarrow B = 1/4$

$$= \int \frac{1/4}{2-t} + \frac{1/4}{2+t}dt$$

$$= \boxed{-\frac{1}{4}\ln|2-t| + \frac{1}{4}\ln|2+t| + C}$$

$$= \boxed{-\frac{1}{4}\ln|2-\sin(x)| + \frac{1}{4}\ln|2+\sin(x)| + C}$$

DON'T FORGET

3. $\int \frac{3}{x-2\sqrt{x}}dx$

$t = \sqrt{x}$

$t^2 = x$

$2t\,dt = dx$

$$\int \frac{3}{t^2-2t}2t\,dt$$

$$\int \frac{6t}{\cancel{t}(t-2)}dt$$

$$= 6\ln|t-2| + C$$

$$= \boxed{6\ln|\sqrt{x}-2| + c}$$

1. $\int e^x \cos(e^x) \sin^3(e^x)\, dx$

$t = e^x$

$dt = e^x dx$

$\int \cos(t) \sin^3(t) dt$

$u = \sin(t)$

$du = \cos(t) dt$

$\int u^3 du$

$\frac{1}{4} u^4 + c$

$= \boxed{\frac{1}{4} \sin^4(e^x) + c}$

How would you *start* these?

1. $\int \tan^3(x)\sec(x)\,dx = \int \tan^2(x)\sec(x)\tan(x)dx$

$= \int (\sec^2(x)-1)\sec(x)\tan(x)dx$

TRIG!

$u = \sec(x)$

$du = \sec(x)\tan(x)dx$

2. $\int x^2\ln(x)\,dx = \frac{1}{3}x^3\ln(x) - \int \frac{1}{3}x^2 dx$

$= \frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3 + C$

BY PARTS!

$u = \ln(x) \quad dv = x^2 dx$

$du = \frac{1}{x}dx \quad v = \frac{1}{3}x^3$

3 $\int x\sqrt{5-x^2}dx = \int x\sqrt{u}\,\frac{1}{-2x}du = -\frac{1}{2}\frac{2}{3}u^{3/2} + C$

$= -\frac{1}{3}(5-x^2)^{3/2} + C$

SUB. (OR TRIG SUB)

$u = 5 - x^2$ ← EASIER

$du = -2x\,dx$

$-\frac{1}{2x}du = dx$

OR

$x = \sqrt{5}\sin(\theta)$

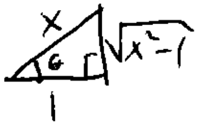
4. $\int \frac{\sqrt{x^2-1}}{x^2}dx = \int \frac{\sqrt{\sec^2\theta - 1}}{\sec^2\theta}\sec\theta\tan\theta\,d\theta = \int \frac{\tan^2\theta}{\sec\theta}d\theta$ OR $\int \frac{\sec^2\theta - 1}{\sec\theta}d\theta$

$= \int \frac{\sin^2\theta}{\cos\theta}d\theta$

TRIG SUB

$x = \sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$



5. $\int \frac{x^2+1}{x^2-2x-3}dx = \int 1 + \frac{2x+4}{x^2-2x-3}dx$

$\frac{2x+4}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$

$x^2-2x-3 \,\overline{)\, x^2+1}$ quotient 1.

$-(x^2-2x-3)$

$2x+4$

DIVIDE, THEN FACTOR DENOM

6. $\int x\tan^{-1}(x)dx = \frac{1}{2}x^2\tan^{-1}(x) - \frac{1}{2}\int \frac{x^2}{x^2+1}dx$

↑ DIVIDE!!

BY PARTS

$u = \tan^{-1}(x) \quad dv = x\,dx$

$du = \frac{1}{x^2+1}dx \quad v = \frac{1}{2}x^2$

7. $\int \frac{d\cancel{x}}{\sqrt{4x^2+8x-12}}dx$

$\sqrt{4(x^2+2x+1-1-3)} = 2\sqrt{(x+1)^2-4}$

$\searrow \int \frac{1}{2\sqrt{4\sec^2\theta - 4}}2\sec\theta\tan\theta\,d\theta$

$= \frac{1}{2}\int \frac{1}{\cancel{2}\cancel{\tan\theta}}\cancel{2}\sec\theta\cancel{\tan\theta}\,d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$

COMPLETE SQUARE

& TRIG SUB!

$x + 1 = 2\sec\theta$